# UNIT-II

# OPERATIONS ON RANDOM VARIABLES

**Expectation (or) Mean:-**

It is denoted as "$E(x)$" (or) "$\bar{x}$" (or) "$\mu_x$"

$X = \{1, 1, 2, 1, 2, 2, 2, 3, 1, 3, 4, 4, 5, 3, 6\}$

$$\text{Mean} = \frac{1+1+2+1+2+2+2+3+1+3+4+4+5+3+6}{15}$$

$$= \frac{1+1+1+1+2+2+2+2+3+3+3+4+4+5+6}{15}$$

$$= \frac{1\times4 + 2\times4 + 3\times3 + 4\times2 + 5\times1 + 6\times1}{15}$$

$$\text{Mean} = 1\underbrace{\left(\frac{4}{15}\right)}_{P(1)} + 2\underbrace{\left(\frac{4}{15}\right)}_{P(2)} + 3\underbrace{\left(\frac{3}{15}\right)}_{P(3)} + 4\underbrace{\left(\frac{2}{15}\right)}_{P(4)} + 5\underbrace{\left(\frac{1}{15}\right)}_{P(5)} + 6\underbrace{\left(\frac{1}{15}\right)}_{P(6)}$$

$$\text{mean} = 1\cdot P(1) + 2\cdot P(2) + 3\cdot P(3) + 4\cdot P(4) + 5\cdot P(5) + 6\cdot P(6)$$

$X = \{x_1, x_2, x_3, x_4\}$

$P(X) = \{P(x_1), P(x_2), P(x_3), P(x_4)\}$

$\text{mean}(x) = x_1\cdot P(x_1) + x_2\cdot P(x_2) + x_3\cdot P(x_3) + x_4\cdot P(x_4)$

Expected value / mean(x) / $\bar{x}$ / $\mu_x$ = $\sum_i x_i P(x_i)$

Continuous Expected value $E[x] = \mu_x = \bar{x} = \int_{-\infty}^{\infty} x f_X(x)\,dx$.

Expected value is represented by $E(x) = \mu_x = \bar{x}$

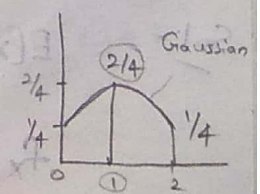
Discrete expected value $\rightarrow E[x] = \sum_i x_i \cdot P(x_i)$

Continuous expected value $\rightarrow E[x] = \int_{-\infty}^{\infty} x f_X(x)\,dx$.

1) Tossing 2 fair coins find the expected value of Random Variable x count number of heads.

Sol:- Sample space $S = \{HH, HT, TH, TT\}$

$x = \{2, 1, 1, 0\}$




$E[x] = \sum_i x_i P(x_i) = 2\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 0\cdot\left(\frac{1}{4}\right)$

$= 2+2+0 / 4 = 1$

2) Find the expected value of a random variable X which performs the sum of faces of two dices.

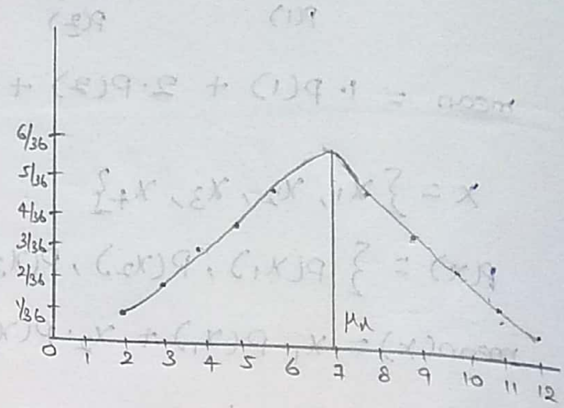
| $D_1 \backslash D_2$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| 4 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| 5 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| 6 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

$X = \{2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 11, 11, 12\}$

$$E[X] = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36}$$

$$= 7$$

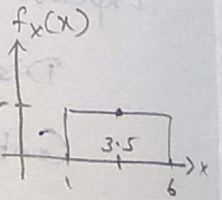



3) Find the expected value of a random variable X whose density function is 1/5 for $1 < x < 6$

Sol:- $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_1^6 x \frac{1}{5}\,dx = \frac{1}{5}\left[\frac{x^2}{2}\right]_1^6 = \frac{1}{10}(36-1)$$

$$= \frac{35}{10} = 3.5$$




4) X is uniformly distributed in the range of 200 to 300 find the expected value of the random variable.

Sol:- $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$f_X(x) = \frac{1}{b-a} \quad \forall\ a < x < b$$

$$f_X(x) = \frac{1}{300-200} = \frac{1}{100} \quad \forall\ 200 < x < 300$$

$$E[x] = \int_{200}^{300} x \cdot \frac{1}{100}\, dx \Rightarrow \frac{1}{100}\left[\frac{x^2}{2}\right]_{200}^{300} = \frac{1}{200}[90000 - 40000]$$

$$= \frac{50000}{200} = 250$$

→ Prove $E[cx] = c\,E[x]$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_x(x)\, dx$$

$$E[cx] = \int_{-\infty}^{\infty} c x \cdot f_x(x)\, dx$$

$$= c \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$E[cx] = c \cdot E[x]$$

## Properties of Expected value :-

1.) Let $c$ is a constant Expected value of 'c' is 'c'.

Proof:- We know $E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$ $\quad E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$

$$E[c] = \int_{-\infty}^{\infty} c \cdot f_x(x)\, dx = c \underbrace{\int_{-\infty}^{\infty} f_x(x)\, dx}_{1}$$

$$= c \cdot 1 = c.$$

2.) Let $a$ is constant and $g(x) = ax$, $E[ax] = a \cdot E[x]$

$x(t) \rightarrow a\,x(t)$ Amplitude scaling

$\rightarrow x(at)$ Time scaling

Proof:- $E[g(x)] = E[ax] = \int_{-\infty}^{\infty} a x\, f_x(x)\, dx$

$$= a \underbrace{\int_{-\infty}^{\infty} x\, f_x(x)\, dx}_{E(x)}$$

$$E[g(x)] = E[ax] = a \cdot E[x]$$

3) Let $a, b$ are constants $g(x) = ax + b$, $E[g(x)] = E[ax+b] = a\,E[x] + b$

$$E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$$

$$E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} ax f_X(x)\,dx + \int_{-\infty}^{\infty} b f_X(x)\,dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{E[X]} + b\underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

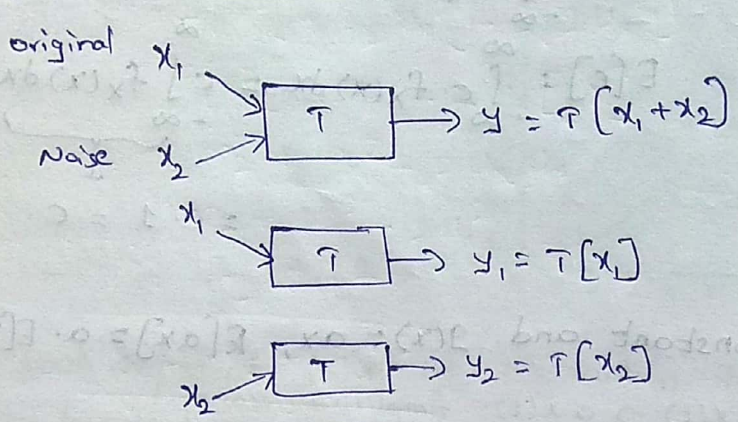
$$E[ax+b] = aE[X] + b$$

4.) If $X>0$, $E[X]>0$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \underbrace{\int_{-\infty}^{0} x f_X(x)\,dx}_{0} + \int_{0}^{\infty} x f_X(x)\,dx$$

$$E(X) = \int_{0}^{\infty} x f_X(x)\,dx > 0.$$

5.) Superposition theorem :-




$$\boxed{y = y_1 + y_2}$$

Response of total signals are equal to the individual signals responses.

Superposition theorem :-

Let $g(x_1)$, $g(x_2)$ are two random variables

$$E[g(x_1) + g(x_2)] = E[g(x_1)] + E[g(x_2)]$$

Let $h(x) = g(x_1) + g(x_2)$ ; $E[h(x)] = E[g(x_1) + g(x_2)]$

$$E[h(x)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx = \int_{-\infty}^{\infty} [g(x_1) + g(x_2)] f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} g(x_1) f_X(x)\,dx + \int_{-\infty}^{\infty} g(x_2) f_X(x)\,dx$$

$E[h(x)] = E[g(x_1)] + E[g(x_2)]$

$E[g(x_1) + g(x_2)] = E[g(x_1)] + E[g(x_2)]$

## Moment:-

① Moment about origin → starting point

② Moment about mean / center moment

$x = \{x_1, x_2, x_3, x_4, x_5\}$ ($x_1$ ↑ origin, $x_3$ ↑ mean)

### ① Moment about origin:-

Also called as $n^{th}$ moment about origin

$$\boxed{m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx}$$

**$0^{th}$ moment about origin:-**

$$m_0 = E[x^0] = E[1] = 1$$

**$1^{st}$ moment about origin:-**

$$m_1 = E[x^1] = E[x] = \text{mean} = \bar{x} = \mu_x$$

**$2^{nd}$ moment about origin:-**

$$m_2 = \underbrace{E}_{\text{mean}}[\underbrace{x^2}_{\text{square}}] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

Root mean square (r.m.s) $= \boxed{\sqrt{m_2} = \sqrt{E(x^2)}}$

**$10^{th}$ moment about origin:-**

$$m_{10} = E[x^{10}] = \int_{-\infty}^{\infty} x^{10} f_X(x)\,dx$$

### ② Moment about mean / central moment

W.K.T, moment about origin $E[x^2] : E[(x-0)^2]$

* $n^{th}$ moment about mean - $[\mu_n]$

$$\mu_n = \boxed{E[(x-\mu_x)^n] = \int_{-\infty}^{\infty} (x-\mu_x)^n f_X(x)\,dx}$$

* $0^{th}$ moment about mean

$n=0 \Rightarrow \mu_n = E[(x-\mu_x)^n]$

$\mu_0 = E[(x-\mu_x)^0] = E[1]$

$$\boxed{\mu_0 = 1}$$

* $1^{st}$ moment about mean $\Rightarrow n=1$

$$\mu_1 = E[(x-\mu_x)^1]$$

$$\mu_1 = \int_{-\infty}^{\infty} (x-\mu_x) f_x(x)\,dx$$

Proof:- $\mu_1 = E[x-\mu_x] = E[x] - E[\mu_x]$ ↗ Constant

$$= \mu_x - \mu_x$$

$$= 0$$

$$\boxed{\mu_1 = 0}$$

* $2^{nd}$ moment about mean $E[(x-\mu_x)^2] = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx$

$2^{nd}$ moment about mean is represented as "variance"

Proof:- $n=2$

$$\mu_2 = E[(x-\mu_x)^2]$$

$$= E[x^2 + \mu_x^2 - 2x\mu_x]$$

$$= E[x^2] + E[\mu_x^2] - E[2\mu_x x]$$

$$= E[x^2] + \mu_x^2 - 2\mu_x E[x] \quad [\because E[cx] = cE[x]]$$

$$= m_2 + \mu_x^2 - 2\mu_x \mu_x$$

$$= m_2 + \mu_x^2 - 2\mu_x^2$$

$$= m_2 - \mu_x^2$$

$$\boxed{\mu_2 = m_2 - (m_1)^2}$$

* $3^{rd}$ moment about mean :- (skew)

$$\mu_3 = E[(x-\mu_x)^3]$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^3 f_x(x)\,dx$$

Skew is used in - Data science / Image processing & computer using / Machine learning.

** Variance :-

$2^{nd}$ moment about mean of a Random variable $\sigma_x^2$

$$\sigma_x^2 = E[(x-\mu_x)^2] = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx$$

Continuous - $\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx$

discrete - $\sigma_x^2 = \sum_i (x_i - \mu_{x_i})^2 P(x_i)$

$$\boxed{\sigma_x^2 = m_2 - (m_1)^2}$$

**Standard deviation :-**

$$\sigma_x = \sqrt{\sigma_x^2}$$

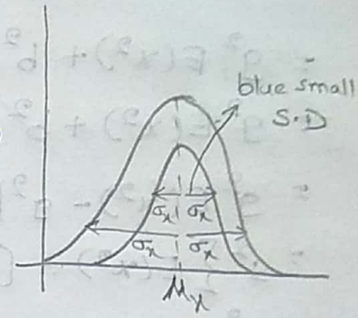
$$= \sqrt{E(x-\mu_x)^2}$$

Standard deviation is a spread of density function about mean.

**Skew coefficient :-**

$$\text{Skew coefficient} = \frac{3^{rd}\text{ moment about mean}}{\text{Cube of standard deviation}}$$

$$= \frac{E\left[(x-\mu_x)^3\right]}{\left[\sqrt{E[(x-\mu_x)^2]}\right]^3}$$




**Properties of variance:-**

① Let X is a R.V, variance of X is represented as var(x), var(c) = 0
constant

**Proof:-** We know,

$$var(x) = m_2 - (m_1)^2$$

$$= E(x^2) - [E(x)]^2$$

$$var(c) = E(c^2) - [E(c)]^2$$

$$= c^2 - (c)^2 = c^2 - c^2 = 0.$$

② Let X is a R.V, its variance is represented as var(x), $var(cx) = c^2 . var(x)$.

**Proof:-** $var(x) = m_2 - (m_1)^2 = E(x^2) - [E(x)]^2$

$$var(cx) = E[(cx)^2] - [E(cx)]^2$$

$$= E[c^2x^2] - [c . E(x)]^2$$

$$= c^2 E[x^2] - c^2 [E(x)]^2$$

$$= c^2 [E(x^2) - [E(x)]^2]$$

$$= c^2 [var(x)]$$

③ If $X$ is a R.V, variance of $X$ is $var(X)$; Let $f = gx+b$ then $var(f) = g^2\, var(x)$. [let $g, b$ are constants]

Proof:-
$$var(x) = E(x^2) - [E(x)]^2$$
$$var(f) = E(f^2) - [E(f)]^2 \qquad [\because f = gx+b]$$
$$= E[(gx+b)^2] - [E(gx+b)]^2$$
$$= E[g^2x^2 + b^2 + 2gbx] - [E(gx + E(b)]^2$$
$$= E(g^2x^2) + E(b^2) + E(2gbx) - \{[E(gx)]^2 + [E(b)]^2 + 2E(gx)\cdot E(b)\}$$
$$= g^2 E(x^2) + b^2 + 2gb\,E(x) - \{g^2[E(x)]^2 + b^2 + 2g\,E(x)\cdot b\}$$
$$= g^2 E(x^2) + \cancel{b^2} + \cancel{2gb\,E(x)} - g^2[E(x)]^2 - \cancel{b^2} - \cancel{2gb\,E(x)}$$
$$= g^2 E(x^2) - g^2[E(x)]^2$$
$$= g^2\{E(x^2) - [E(x)]^2\}$$
$$= g^2\, var(x).$$

## Moment generating functions:-

Classified into two types

① Moment generating function (MGF)

② Characteristic function (CF)

### ① Moment generating function (MGF):-

Statement:- Let $X$ is a Random Variable, its MGF is represented as $M_X(t)$ ? Latent variable.

$$\boxed{M_X(t) = E[e^{xt}] = \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx.}$$

** Note:- MGF used to generate moment about origin.

Generation of moments from MGF:-

$$M_X(\underset{\text{Latent variable}}{t}) = E[e^{xt}]$$
$$= E\left[\frac{(xt)^0}{0!} + \frac{(xt)^1}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \cdots + \frac{(xt)^n}{n!} + \cdots\right]$$
$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right]$$

$$M_X(t) = E(1) + E\left[\frac{tx}{1!}\right] + E\left[\frac{t^2x^2}{2!}\right] + E\left[\frac{t^3x^3}{3!}\right] + \cdots + E\left[\frac{t^nx^n}{n!}\right] + \cdots$$

$$M_X(t) = 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots + \frac{t^n}{n!}E(x^n) + \cdots$$

$0^{th}$ moment from MGF is calculated by submitting $t=0$.

$$m_0 = M_X(t)/_{t=0} = 1$$

$1^{st}$ moment about origin.

$$\frac{d}{dt}M_X(t) = 0 + 1\cdot E(x) + \frac{2t}{2!}E(x^2) + \frac{3t^2}{3!}E(x^3) + \cdots + \frac{nt^{n-1}}{n!}E(x^n) + \cdots$$

Let $t=0$

$$\frac{d}{dt}M_X(t)/_{t=0} = E(x) = m_1$$

$2^{nd}$ moment about origin :-

$$\frac{d^2}{dt^2}M_X(t) = 0 + 0 + \frac{2}{2!}E(x^2) + \frac{3\cdot 2\cdot t}{3!}E(x^3) + \cdots + \frac{n(n-1)t^{n-2}}{n!}E[x^n]$$

Sub $t=0$

$$\frac{d^2}{dt^2}M_X(t)/_{t=0} = 0 + 0 + E(x^2) + 0 + 0 + \cdots$$
$$= E(x^2) = m_2$$

$n^{th}$ moment about origin using MGF :-

$$\boxed{\frac{d^n}{dt^n}M_X(t)/_{t=0} = m_n}$$

MGF and Properties

MGF :- $M_X(t) = E[e^{xt}]$

① Let $x$ is a Random variable $f_X(x)$ is the density function, its MGF is $M_X(t)$, if $t=0$ then $M_X(0) = 1$.

Proof:- $M_X(t) = E[e^{xt}]$

Sub $t=0$ in MGF

$$M_X(0) = E[e^{x(0)}]$$

$\therefore M_x(0) = E[e^0] \Rightarrow E[1] \Rightarrow 1$

$M_x(0) = 1$

② Let $x$ is a Random variable, $f_x(x)$ is the density function. MGF of $x$ is $M_x(t)$, $y$ is the another random variable $y = ax$ then $M_y(t) = M_x(at)$ [Note:- $a$ is constant]

Proof:-

$$M_x(t) = E[e^{xt}]$$

$$M_y(t) = E[e^{yt}] \qquad y = ax \qquad y = ax$$

variable

$\to$ Latent variable

$$= E[e^{axt}] = E[e^{x(at)}] \qquad = M_x(at)$$

** ③ Let $x$ is a R.V, $M_x(t)$ is the MGF of $x$, Let $y$ is another R.V, $y = ax+b$, $M_y(t) = e^{bt} \cdot M_x(at)$.

Proof:-

$$M_x(t) = E[e^{xt}]$$

$$M_y(t) = E[e^{yt}] \qquad y = ax+b$$

$e^{m+n} = e^m \cdot e^n$

$$= E[e^{(ax+b)t}] = E[e^{axt+bt}] = E[e^{axt} \cdot e^{bt}]$$

$$M_y(t) = e^{bt} E[e^{axt}] = e^{bt} \cdot M_x(at).$$

④ Let $x$ is a R.V, its MGF is $M_x(t)$, $y = \frac{ax+b}{d}$, $a, b, d$ are constants $M_y(t) = e^{bt/d} \cdot M_x\left(\frac{at}{d}\right)$

Proof:-

$$M_x(t) = E[e^{xt}].$$

$$M_y(t) = E[e^{yt}] \qquad y = \frac{ax+b}{d}$$

$$= E\left[e^{\left(\frac{ax+b}{d}\right)t}\right] = E\left[e^{\frac{ax}{d}t + \frac{b}{d}t}\right]$$

$$= E\left[e^{\frac{ax}{d}t} \cdot e^{\frac{b}{d}t}\right]$$

$$= e^{\frac{b}{d}t} E\left[e^{\frac{ax}{d}t}\right]$$

$$= e^{\frac{bt}{d}} \cdot M_x\left(\frac{at}{d}\right)$$

Convergence:- The system that works with bounded input and bounded output [stable system]

Convergence of MGF:-

$$M_X(t) = E[e^{xt}] = \int_{-\infty}^{\infty} e^{xt} f_X(x)dx$$

$$|M_X(t)| = \left| \int_{-\infty}^{\infty} e^{xt} f_X(x) \cdot dx \right|$$

$2.3 \quad 2.3 = 6$
$|ab| = |a| \cdot |b|$
$-2 \cdot 3 = 6 \quad |-2| \cdot |3| = 6$

$$\div \int_{-\infty}^{\infty} |e^{xt} f_X(x)| dx$$

$$= \int_{-\infty}^{\infty} |e^{xt}| |f_X(x)| dx.$$

$$\boxed{\int_{-\infty}^{\infty} f_X(x)dx = 1}$$

R.V

latent variable

$e^{xt}$ → play role that decide the stability of moment

→ MGF fail to get stable moments due to it depends on Latent variable 't'.

② Characteristic function:-

→ Characteristic function is mainly used to overcome the limitations of MGF.

Representation of characteristic function:-

$$\phi_X(j\omega) = E[e^{j\omega x}] \qquad M_X(t) = E[e^{xt}]$$

→ characteristic function is also used to extract required moments

Using Taylor's series

$$\phi_X(j\omega) = E\left[\frac{(j\omega x)^0}{0!} + \frac{(j\omega x)^1}{1!} + \frac{(j\omega x)^2}{2!} + \dots + \frac{(j\omega x)^n}{n!} + \dots\right]$$

$$= E\left[1 + \frac{(j\omega)^1 x}{1!} + \frac{(j\omega)^2 x^2}{2!} + \dots + \frac{(j\omega)^n x^n}{n!} + \dots\right]$$

$$= E[1] + E\left[\frac{j\omega x}{1!}\right] + E\left[\frac{(j\omega)^2 x^2}{2!}\right] + \dots + E\left[\frac{(j\omega)^n x^n}{n!}\right] + \dots$$

$$\phi_X(\omega) = 1 + j\omega E[x] + (j\omega)^2 E\left[\frac{x^2}{2!}\right] + \dots + (j\omega)^n E\left[\frac{x^n}{n!}\right] + \dots$$

Extracting Moments

* $m_0 = \phi_X(\omega)/_{\omega=0}$

* $m_1$ = 1st moment about origin

$$\frac{d}{d\omega}\phi_X(\omega) = 0 + jE(x) + j^2 2\omega \frac{E[x^2]}{2!} + \dots + j^n n\omega^{n-1} \frac{E[x^n]}{n!} + \dots$$

Sub $\omega = 0$

$$\frac{d}{d\omega} \phi_X(\omega) / \omega=0 = j E(x)$$

$$E(x) = \frac{1}{j} \frac{d}{d\omega} \phi_X(\omega) / \omega=0 = -j \frac{d}{d\omega} \phi_X(\omega) / \omega=0$$

* $2^{nd}$ moment about origin :-

$$\frac{d^2}{d\omega^2} \phi_X(\omega) = 0 + 0 + (j)^2 2 \frac{E[x^2]}{2!} + \cdots + (j)^n n(n-1)\omega^{n-2} \frac{E[x^n]}{n!} +$$

Sub $\omega = 0$

$$\frac{d^2}{d\omega^2} \phi_X(\omega) /_{\omega=0} = (j)^2 E[x^2]$$

$$E(x^2) = \frac{1}{(j)^2} \frac{d^2}{d\omega^2} \phi_X(\omega) / \omega=0$$

$(-j)^2 = (-1 \cdot j)^2$

$= (-1)^2 \cdot (j)^2$

$= 1 \cdot -1 = -1$

$$E(x^2) = -\frac{d^2}{d\omega^2} \phi_X(\omega) / \omega=0 = (-j)^2 \frac{d^2}{d\omega^2} \phi_X(\omega)$$

* $n^{th}$ moment with respect to origin

$$E[x^n] = (-j)^n \frac{d^n}{d\omega^n} \phi_X(\omega)$$

Convergence of C.F :-

$$\phi_X(\omega) = E(e^{j\omega x}) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx.$$

$$|\phi_X(\omega)| = \left| \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx \right| = \int_{-\infty}^{\infty} |e^{j\omega x} \cdot f_X(x)| dx.$$

$$= \int_{-\infty}^{\infty} |e^{j\omega x}| \cdot |f_X(x)| dx$$

$\int_{-\infty}^{\infty} f_X(x) dx = 1$

$$|e^{j\omega x}| = |\cos \omega x + j \sin \omega x|$$

$e^{jx} = \cos x + j \sin x$.

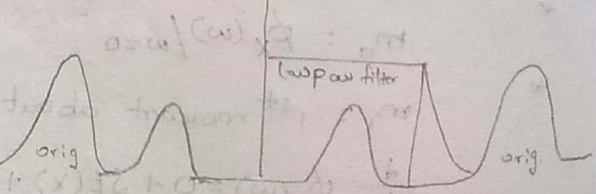
$$= \sqrt{(\cos \omega x)^2 + (\sin \omega x)^2}$$

$0 \le |H(x)| \le 1$

$$= \sqrt{1} = 1$$

$$\phi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx. = \int_{-\infty}^{\infty} f(x) e^{j\omega x} dx.$$

$$F.T[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{j\omega t} dt$$



Properties of characteristic function :

① Let $X$ is the R.V, its C.F is $\phi_X(\omega)$, if $\omega=0$, $\phi_X(\omega)=1$.

Proof:- $\phi_X(\omega) = E[e^{j\omega x}]$

$\omega=0$ $\phi_X(0) = E[e^{j(0)\cdot x}] = E[1] = 1$

② Let $X$ is a R.v. $\phi_X(\omega)$ is char function, $Y = aX$

$\phi_Y(\omega) = \phi_X(a\omega)$.

Proof:- $\phi_X(\omega) = E[e^{j\omega x}]$

$\phi_Y(\omega) = E[e^{j\omega Y}]$ $\quad y = ax$.

$= E[e^{j\omega(ax)}]$

$= E[e^{a(j\omega x)}] = E[e^{j\omega a \cdot x}] = \phi_X(\omega a)$.

$\phi_Y(\omega) = \phi_X(a\omega)$

③ Let $X$ is a R.V, $\phi_X(\omega)$ is the C.F, $Y = ax+b$,

$\phi_Y(\omega) = \phi_X(a\omega)\cdot e^{jb\omega}$

Proof:- $\phi_X(\omega) = E[e^{j\omega x}]$

$\phi_Y(\omega) = E[e^{j\omega y}]$ $\quad y = ax+b$

$\phi_Y(\omega) = E[e^{j\omega(ax+b)}] = E[e^{j\omega ax}\cdot e^{j\omega b}]$

$= e^{j\omega b}\cdot E[e^{j\omega ax}]$

$= e^{j\omega b}\cdot \phi_X(\omega a)$

④ $X$ is a R.V $\phi_X(\omega)$ is the C.F, $y = \frac{ax+b}{d}$ then

$\phi_Y(\omega) = e^{\frac{j\omega b}{d}}\cdot \phi_X\left(\frac{\omega a}{d}\right)$

Proof:- $\phi_X(\omega) = E[e^{j\omega x}]$

$\phi_Y(\omega) = E[e^{j\omega y}]$ $\quad y = \frac{ax+b}{d}$

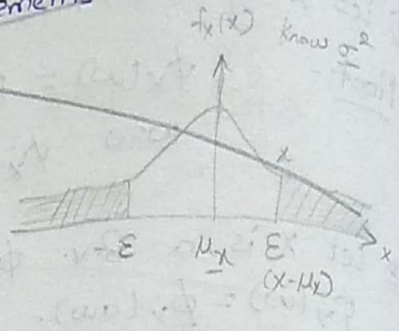
$= E\left[e^{j\omega\left(\frac{ax+b}{d}\right)}\right] = E\left[e^{\frac{j\omega ax}{d}}\cdot e^{\frac{j\omega b}{d}}\right]$

$= e^{\frac{j\omega b}{d}}\cdot E\left[e^{\frac{j\omega ax}{d}}\right] = e^{\frac{j\omega b}{d}}\cdot \phi_X\left(\frac{\omega a}{d}\right)$

***
## Center limit theorem :-

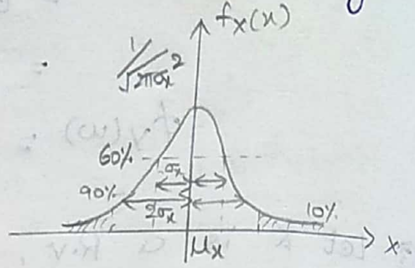
Statement :- Finding the probability of elements beyond the shaded Region

$$P[|x-\mu_x| \geq \epsilon] \leq \frac{\sigma_x^2}{\epsilon^2}$$




***
## Chebyshev's inequality :-

Let X be a random variable having finite mean $\mu_x$, and finite standard deviation $\sigma_x$ it provide how much of data fall in K standard deviations is calculated using Chebysheve inequality.




$$P(|x-\mu_x| \geq \sigma_x) \leq \frac{\sigma_x^2}{\epsilon_x^2}$$

Proof :- $f_x(x)$ is the density function, mean $\mu_x$, standard deviation $\sigma_x$

Varience of a density function :-

i.e., 2nd moment about mean

$$E[(x-\mu_x)^2] = \sigma_x^2$$

Variance of total density function

$$\sigma_x^2 = E[(x-\mu_x)^2] = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx \longrightarrow (1)$$

Variance of shaded region $\sigma_{shaded}^2 = \int_{|x-\mu_x| \geq \epsilon} (x-\mu_x)^2 f_x(x)\,dx \rightarrow (2)$

$$\sigma_{shaded}^2 = \int_{|x-\mu_x| \geq \epsilon} \epsilon^2 f_x(x)\,dx \longrightarrow (3)$$

we know total varience $\geq$ shaded Region variance

$$\sigma_x^2 \geq \sigma_{shaded}^2$$

$$\geq \int_{|x-\mu_x| \geq \epsilon} \epsilon^2 f_x(x)\,dx$$

$[P(|x-\mu_x| \geq \epsilon)]$

$$\sigma_x^2 \geq \epsilon^2 \left[ P(|x-\mu_x| \geq \epsilon) \right]$$

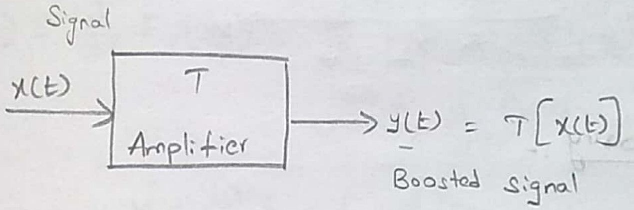
$$\boxed{P\left[|x-\mu_x| \geq \epsilon\right] \leq \frac{\sigma_x^2}{\epsilon_x^2}}$$

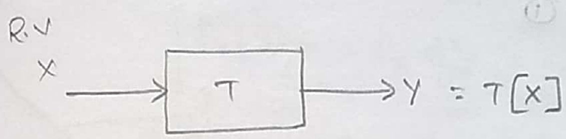
$$\epsilon_x = k\sigma_x$$

$$P\left[|x-\mu_x| \geq k\sigma_x\right] \leq \frac{\sigma_x^2}{k^2 \sigma_x^2}$$

$$\boxed{P\left[|x-\mu_x| \geq k\sigma_x\right] \leq \frac{1}{k^2}}$$

**Transformation :-**




Classified into two types

① Monotonic transformation

② Non monotonic transformation

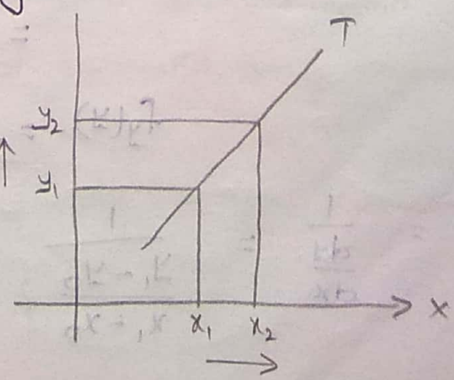
1) **Monotonical Transformation :-** One Random variable X map to one outcome at Y.

Classified into two types

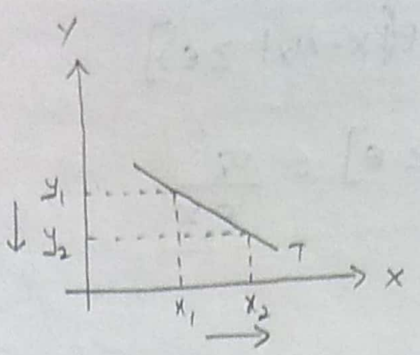
1) Monotonically increasing

2) Monotonically decreasing.

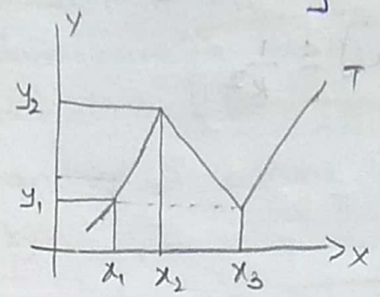
**Monotonically increasing :-**



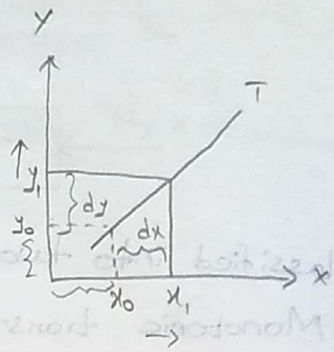
**Monotonically decreasing :-**




2) **Nonmonotonic Transformation :-** Many to one mapping




**Monotonically Increasing :-**




$$Y = T[x] \rightarrow ①$$

$$y_0 = T[x_0]$$

$$x_0 = T^{-1}[y_0] \rightarrow ②$$

Probability upto $y_0$

$$P(Y \leq y_0) = P(X \leq x_0)$$

$$F_Y(y_0) = P(X \leq \underbrace{T^{-1}(y_0)}_{a})$$

$\frac{dx}{dy}$ = change in the x w.r.t change in the y.

$$= F_X(a) = F_X(T^{-1}(y_0))$$

$$F_Y(y_0) = F_X(T^{-1}(y_0))$$

Distribution function of Y upto $y_0$

$$f_Y(y_0) = \frac{d}{dy}[F_Y(y_0)] = \frac{d}{dy}[F_X(T^{-1}(y_0))]$$

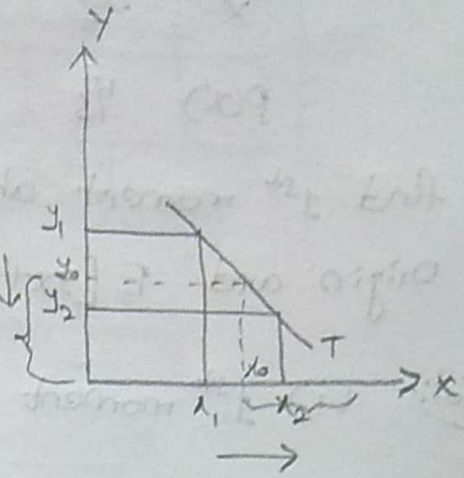
$$= f_X \frac{d}{dy}[T^{-1}(y_0)]$$

$$f_Y(y) = f_X \frac{dx_0}{dy} = f_X\left[\frac{dx}{dy}\right]$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\underbrace{\frac{y_1 - y_0}{x_1 - x_0}}_{\text{slope}}}$$

$$f_y(y) = \frac{f_x}{m} \qquad m \to \text{slope}$$

**Montonically decreasing**




$$Y = T[x] \Rightarrow y_0 = T[x_0]$$

$$x_0 = T^{-1}[y_0]$$

$$P(Y \le y_0) = P(X > x_0)$$

$$= 1 - P(X \le x_0)$$

$$F_Y(y_0) = 1 - P(X \le \underbrace{T^{-1}(y_0)}_{a})$$

$$F_Y(y_0) = 1 - F_X(a) = 1 - F_X(T^{-1}(y_0))$$

$$f_y(y_0) = \frac{d}{dY} F_Y(y_0) = \frac{d}{dY}\left[1 - F_X[T^{-1}(y_0)]\right]$$

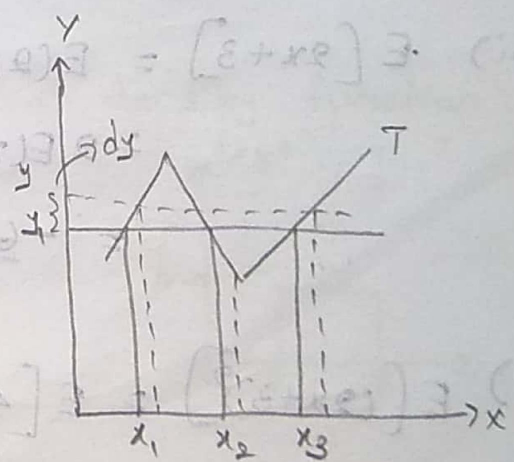
$$= 0 - f_X \frac{d}{dY}\left[T^{-1}(y_0)\right]$$

$$f_y(y_0) = -f_X \frac{dx_0}{dY}$$

$$f_Y(y) = -f_X \frac{dX}{dY} = f_X\left(\frac{-1}{m}\right)$$

Montonically ~~decreasing~~ $|f_Y(y)| = \left|f_x\left(\frac{dX}{dy}\right)\right|$

**Non-monotonic :-**




$$\cancel{f_Y(y)dy \neq f_X(x_1}$$

$$f_Y(y) = f_{X}\left(\frac{dx_1}{dy}\right) + f_X\left(\frac{dx_2}{dy}\right) + f_X\left(\frac{dx_3}{dy}\right)$$

$$= f_X\left[\frac{1}{m_1}\right] + f_X\left[\frac{1}{m_2}\right] + f_X\left[\frac{1}{m_3}\right]$$

1) Let X is a R.v with probability mass function given in the table

| X | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| P(X) | 1/5 | 2/5 | 1/10 | 1/10 | 1/5 |

find 1st moment about origin and 2nd moment about origin and $E[2x+3]$, $E[(2x+3)^2]$

Sol:- (i) 1st moment about origin = mean ($\mu_x$) = $E(x) = \sum x_i P(x_i)$

$= (-2)\left(\frac{1}{5}\right) + (-1)\left(\frac{2}{5}\right) + (0)\left(\frac{1}{10}\right) + (1)\left(\frac{1}{10}\right) + (2)\left(\frac{1}{5}\right)$

$= -\frac{2}{5} - \cancel{\frac{2}{5}} + \frac{1}{10} + \cancel{\frac{2}{5}}$

$= \frac{-4+1}{10} = \frac{-3}{10}$.

(ii) 2nd moment about origin = $E(x^2) = \sum x_i^2 p(x_i)$

$= (-2)^2\left(\frac{1}{5}\right) + (-1)^2\left(\frac{2}{5}\right) + (0)^2\left(\frac{1}{10}\right) + (1)^2\left(\frac{1}{10}\right) + (2)^2\left(\frac{1}{5}\right)$

$= \frac{4}{5} + \frac{2}{5} + \frac{1}{10} + \frac{4}{5} = 2 + \frac{1}{10}$

$= \frac{21}{10}$

(iii) $E[2x+3] = E(2x) + E(3)$

$= 2E(x) + 3 = 2\left(-\frac{3}{10}\right) + 3$

$= \frac{-6+30}{10} = \frac{24}{10}$.

(iv) $E[(2x+3)^2] = E[4x^2 + 9 + 12x] = E[4x^2] + E[9] + E[12x]$

$= 4E[x^2] + 9 + 12E(x)$

$= 4\left(\frac{21}{10}\right) + 9 + 12\left(-\frac{3}{10}\right)$

$= \frac{84-36}{10} + 9 = \frac{48}{10} + 9 = \frac{48+90}{10}$

$= \frac{138}{10}$

$\begin{array}{r} 7 \\ 84 \\ -36 \\ \hline 48 \end{array}$

2) A R.V X is defined with a density function $f_X(x) = 3x$ $\forall\ 0 \le x \le 4$ find mean and variance of the R.V

sol: i) Mean $E(x) = \mu_x = \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx$

$$= \int_0^4 x \cdot 3x\,dx = \int_0^4 3x^2\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^4 = 4^3 - 0 = 64$$

(ii) Variance :- $2^{nd}$ moment about mean

$$E\left[(x-\mu_x)^2\right] = m_2 - (m_1)^2$$

$m_2 \rightarrow 2^{nd}$ moment about origin,

$m_1 \rightarrow 1^{st}$ moment about origin.

$$m_2 = E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_0^4 x^2 \cdot 3x \cdot dx$$

$64 \times 3 = 192$

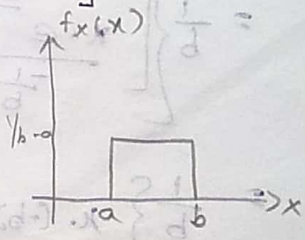
$$= 3\left[\frac{x^4}{4}\right]_0^4 = \frac{3}{\cancel{4}}(4^{\cancel{4}} - 0)$$

$$= 192$$

$$\sigma_x^2 = m_2 - (m_1)^2 = 192 - (64)^2$$

$$\sigma_x^2 = -3904$$

3) Find mean and variance of uniform density function.




$$f_x(x) = \frac{1}{b-a} \quad \forall \quad a \le x \le b$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2(b-a)}(b^2 - a^2)$$

$$= \frac{1}{2\cancel{(b-a)}}(b+a)\cancel{(b-a)} = \frac{b+a}{2}$$

$$m_1 = E(x) = \frac{b+a}{2}$$

Variance : $E\left[(x-\mu_x)^2\right] = m_2 - (m_1)^2$

$$m_2 = E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}(b^3-a^3) = \frac{(b-a)(b^2+a^2+ab)}{3(b-a)}$$

$$m_2 = \frac{b^2+a^2+ab}{3}$$

$$\sigma_x^2 = m_2 - (m_1)^2$$

$$= \frac{b^2+a^2+ab}{3} - \left[\frac{b+a}{2}\right]^2 = \frac{b^2+a^2+ab}{3} - \frac{1}{4}\left[b^2+a^2+2ab\right]$$

$$= \frac{4a^2+4b^2+4ab-3b^2-3a^2-6ab}{12} = \frac{a^2+b^2-2ab}{12}$$

$$\sigma_x^2 = \frac{(a-b)^2}{12}$$

4) Find the mean and variance of exponential density function

Sol:-

$$f_X(x) = \frac{1}{b}\cdot e^{-\frac{(x-a)}{b}} \quad \forall \quad x > a$$

$$E(X) = \int_{-\infty}^{\infty} x.f_X(x)\,dx = \int_a^{\infty} x.\frac{1}{b}e^{-\frac{(x-a)}{b}}\,dx$$

$$= \frac{1}{b}\int_a^{\infty} x.e^{-\frac{(x-a)}{b}}\,dx.$$

$$\left[\int uv = u\int v\,dx - \int u'\int v\,dx\right]$$

$$= \frac{1}{b}\left\{\left[\frac{x.e^{-\frac{(x-a)}{b}}}{-\frac{1}{b}}\right]_a^{\infty} - \int_a^{\infty} 1.\frac{e^{-\frac{(x-a)}{b}}}{-1/b}\right\}$$

$$= \frac{1}{b}\left\{x.(-b)[0-1] - \int_a^{\infty} -b\,e^{-\frac{(x-a)}{b}}\,dx\right\}$$

$$= \frac{1}{b}\left\{x.b + b\int_a^{\infty} e^{-\frac{(x-a)}{b}}\,dx\right\}$$

$$= \frac{1}{b}\left\{bx + b\left[\frac{e^{-\frac{(x-a)}{b}}}{-1/b}\right]_a^{\infty}\right\}$$

$$= \frac{1}{b}\left\{bx + b[-b(0-1)]\right\} = \frac{1}{b}\left\{bx + b^2\right\}$$

$$= x + b$$

Variance $E\left[(x-\mu_x)^2\right] = m_2 - (m_1)^2$

$$m_2 = E(x^2) = \int_a^\infty x^2 \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$$

## Pascal triangle and relation with binomial

Row - 0    1    $a, b$   $(a+b)^0 = 1$

Row - 1    1 1    $(a+b)^1 = a+b$

Row - 2    1 2 1    $(a+b)^2 = a^2+2ab+b^2$

Row - 3    1 3 3 1    $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

Row - 4    1 4 6 4 1

Bernaulli trail :

$${}^nc_{x=0,1}\, p^x (1-p)^{n-x}$$

$n=1 \Rightarrow 0,1$

$${}^1c_0\, p^0 (1-p)^{1-0} + {}^1c_1\, p^1 (1-p)^{1-1}$$

$p = a$

$1-p = b$

$$1.1.(1-p) + 1.p^1(1-p)^0$$

$$1.1.b + 1.a.1 = b+a$$

Let $n=2$, Row 2:-

$n=2$, $x=0,1,2$

$${}^2c_0\, a^0\, b^{2-0} + {}^2c_1\, a^1 . b^{2-1} + {}^2c_2\, a^2 . b^{2-2}$$

$$= 1.b^2 + 2.a.b + 1.a^2 = a^2+2ab+b^2$$

$n^{th}$ Row Equation :-

$n=n$, $x=0,1,2,\ldots\ldots n$

$$= {}^nc_0\, a^0 b^{n-0} + {}^nc_1\, a^1 b^{n-1} + {}^nc_2\, a^2 b^{n-2} + \ldots\ldots + {}^nc_n\, a^n b^{n-n}$$

$$n^{th} \text{ row} = \sum_{x=0}^{n} {}^nc_x\, a^x b^{n-x} = (a+b)^n$$

From pascal triangle ;

$n^{th}$ row contains $n-1$ coefficients

$\therefore (a+b)^{n-1}$

## Mean and variance of binomial density function

$$P(x) = {}^nc_x\, p^x (1-p)^{n-x} \rightarrow \text{Binomial density}$$

Assumptions $\left\{ \begin{array}{l} p \rightarrow \text{True outcome} \quad (1-p) \text{ False outcome} \\ p+(1-p) = 1 \Rightarrow p+q = 1 \end{array} \right.$

$$\text{Mean } E(x) = \mu_x = \sum_{x=0}^{n} x.P(x) = \sum_{x=0}^{n} x.\,{}^nc_x\, p^x q^{n-x} \rightarrow (1)$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \cdot \frac{n!}{\cancel{x}(x-1)!\,(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!}\, p^x . q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x+1-1)!} \cdot p . p^{x-1} q^{n-x+1-1}$$

$$= n.p \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-1-(x-1))!}\, p^{x-1} q^{n-1-(x-1)}$$

$x-1 = \omega$, if $x=1$, $\omega = 0$

if $x = n$, $\omega = n-1$

$$= n.p \sum_{\omega=0}^{n-1} \frac{(n-1)!}{\omega!\,(n-1-\omega)!}\, p^{\omega} q^{n-1-\omega}$$

$$= n.p \underbrace{\sum_{\omega=0}^{n-1} {}^{(n-1)}c_{\omega}\, p^{\omega} q^{(n-1)-\omega}}_{n^{th} \text{ row} = \sum_{x=0}^{n} {}^{n}c_x a^x b^{n-x}} \longrightarrow (n-1)^{th} \text{ row}$$

$$= np(p+q)^{n-1} = np(1)^{n-1} = n.p$$

<u>Variance:-</u> $\sigma_x^2 = E(x-\mu_x)^2 = m_2 - (m_1)^2$

$$m_2 = E(x^2) = \sum_{x=0}^{n} x^2 . p(x)$$

$$= \sum_{x=0}^{n} x^2 . {}^{n}c_x\, p^x (q)^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \cdot \frac{n!}{x!\,(n-x)!}\, p^x (q)^{n-x}$$

$x^2 + x - x$

$(x^2 - x + x)$

$$= \sum_{x=0}^{n} \underbrace{(x^2 - x + x)}_{(a+b)} \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x) \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x} + \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

$1^{st}$ term:-

$$\sum_{x=0}^{n} (x^2-x) \frac{n!}{x!(n-x)!} p^x q^{n-x} = \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{n!}{\cancel{x(x-1)}(x-2)!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!(n-x+2-2)!} p^2 p^{x-2} q^{n-x+2-2}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-2-(x-2))!} p^{x-2} q^{n-2-(x-2)}$$

$$= n(n-1)p^2 (p+q)^{n-2}$$

$1^{st}$ term $= n(n-1)p^2 = (n^2-n)p^2 = p^2n^2 - p^2n$

$2^{nd}$ term $= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x} = np$

$m_2 = 1^{st}$ term $+ 2^{nd}$ term

$= p^2n^2 - p^2n + np$

$$\sigma_x^2 = m_2 - (m_1)^2 = \cancel{p^2n^2} - p^2n + np - \cancel{(np)^2}$$

$$= pn - p^2n$$

$$= pn(1-p)$$

$$\sigma_x^2 = pnq.$$

Mean and variance of poission's density function:-

$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ = probability mass function

mean value $E(X) = \sum_{x=0}^{\infty} x \cdot P(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$

$$= \sum_{x=0}^{\infty} \cancel{x} \cdot \frac{e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda\left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \cancel{e^{-\lambda}}\cdot\lambda\cdot\cancel{e^{\lambda}} = \lambda$$

$$\boxed{E(x) = \lambda}$$

Variance $\sigma_x^2 = E[(x-\mu_x)^2] = m_2 - (m_1)^2$

$$m_2 = E(x^2) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$x^2 = x^2 + x - x$

$x^2 = x(x-1) + x$

$$= \sum_{x=0}^{\infty}[x(x-1)+x]\cdot\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \underbrace{\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}}_{\text{Term-1}} + \underbrace{\sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x!}}_{\text{Term-II}}$$

Term-I :-

$$\sum_{x=0}^{\infty} \frac{\cancel{x(x-1)}\, e^{-\lambda}\lambda^x}{\cancel{x(x-1)}(x-2)!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} \Rightarrow \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^2\lambda^{x-2}}{(x-2)!} \Rightarrow e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}$$

$x-2 = t \Rightarrow x = 2 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$= e^{-\lambda}\lambda^2 \sum_{t=0}^{\infty}\frac{\lambda^t}{t!} = e^{-\lambda}\lambda^2\Big[\underbrace{\frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^\infty}{\infty!}}_{e^{\lambda}}\Big]$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

II-Term :- $\sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x!} = \lambda$

$m_2$ = I-Term + II-Term $= \lambda^2 + \lambda$

Variance $\sigma_x^2 = m_2 - (m_1)^2 = \lambda^2 + \lambda - (\lambda)^2$

$$= \lambda$$

## **Mean and variance of Gaussian density function:-

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \quad \forall \quad -\infty \text{ to } \infty$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \cdot dx.$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\infty} x \cdot e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \, dx.$$

$$\frac{x-\mu_x}{\sigma_x} = t$$

$$x - \mu_x = t\sigma_x \Rightarrow x = \sigma_x t + \mu_x$$

$$dx = \sigma_x \, dt$$

$$x = -\infty \Rightarrow t = -\infty$$

$$x = \infty \Rightarrow t = \infty$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\infty} (\sigma_x t + \mu_x) e^{-t^2/2} \sigma_x \, dt.$$

$$= \frac{1}{\sqrt{2\pi\cancel{\sigma_x^2}}} \cancel{\sigma_x} \int_{-\infty}^{\infty} (\sigma_x t + \mu_x) e^{-t^2/2} \, dt.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \sigma_x t e^{-t^2/2} dt + \int_{-\infty}^{\infty} \mu_x \cdot e^{-t^2/2} dt \right\}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x t \cdot e^{-t^2/2} dt}_{\text{Term-I}} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu_x \cdot e^{-t^2/2} dt}_{\text{Term-II}}$$

Term-I $\quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x t \cdot e^{-t^2/2} dt = \frac{\sigma_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t \cdot e^{-t^2/2} dt. \quad \int_{-\infty}^{0} + \int_{0}^{\infty}$

$\to$ odd function $= 0$.

Term - II $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mu_x \cdot e^{-t^2/2} dt$

$$= \frac{\mu_x}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2} dt = \frac{\mu_x}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^0 \cdot e^{-t^2/2} dt = 1$$

$$= \mu_x \cdot 1 = \mu_x \quad \cancel{\text{Mean}}$$

mean $E(x)$ of Gaussian

$$\text{Term - I} + \text{Term - II} = 0 + \mu_x = \mu_x \quad \boxed{\text{mean} = \mu_x}$$

Variance :- $\sigma_x^2 \; E[(x-\mu_x)^2] = m_2 - (m_1)^2$

$$m_2 = E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}\cancel{\sigma_x}}\int_{-\infty}^{\infty} (\sigma_x t + \mu_x)^2 e^{-t^2/2} \cancel{\sigma_x}\, dt$$

$\frac{x-\mu_x}{\sigma_x} = t$

$x = \sigma_x t + \mu_x$

$dx = \sigma_x dt$.

$x = -\infty \Rightarrow t = -\infty$

$x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma_x^2 t^2 + \mu_x^2 + 2\sigma_x t \cdot \mu_x) e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sigma_x^2 t^2 \cdot e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mu_x^2 \cdot e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sigma_x \mu_x t e^{-t^2/2} dt$$

$$= \underbrace{\frac{\sigma_x^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 \cdot e^{-t^2/2} dt}_{\to 1} + \underbrace{\frac{\mu_x^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2} dt}_{\to 1} + \underbrace{\frac{2\sigma_x \mu_x}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t \cdot e^{-t^2/2} dt}_{\to 0}$$

$$= \sigma_x^2 + \mu_x^2.$$

Variance $\sigma_x^2 = m_2 - (m_1)^2 = \sigma_x^2 + \cancel{\mu_x^2} - \cancel{(\mu_x)^2} = \sigma_x^2$

$$\boxed{\text{Variance} = \sigma_x^2}$$

## Mean and variance of Rayleigh density function

$$f_X(x) = \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \quad \forall\ x > a$$

Mean $\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_a^{\infty} x \cdot \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}\, dx$$

let $\frac{x-a}{b} = t$

$x - a = bt$

$x = bt + a$

$dx = b\,dt$

L.L :- $x = a$ ; $t = 0$

U.L :- $x = \infty$, $t = \infty$

$$= 2\int_0^{\infty} (bt+a)\, t \cdot e^{-bt^2}\, b\,dt$$

$$= 2b\left[\int_0^{\infty} (bt^2 + at)\, e^{-bt^2}\, dt\right]$$

$$= 2b\left[\int_0^{\infty} bt^2 e^{-bt^2}\, dt + \int_0^{\infty} at e^{-bt^2}\, dt\right]$$

$$= 2b(b)\int_0^{\infty} t^2 e^{-bt^2}\, dt + 2b(a)\int_0^{\infty} t \cdot e^{-bt^2}\, dt$$

(1) even function (0) odd function

$$= 2b^2(1)$$

$$E(X) = 2b^2$$

Variance:- $\sigma_X^2 = E[(X-\mu_X)^2] = m_2 - (m_1)^2$

$$m_2 = E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_a^{\infty} x^2\, \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}\, dx$$

let $\frac{x-a}{b} = t$

$x - a = bt$

$x = bt + a$

$dx = b\,dt$

L.L :- $x = a$; $t = 0$

U.L :- $x = \infty$; $t = \infty$

$$= 2\int_0^{\infty} (bt+a)^2\, t\, e^{-bt^2}\, b\,dt$$

$$= 2b\int_0^{\infty} \left[b^2t^2 + a^2 + 2abt\right] t\, e^{-bt^2}\, dt$$

$$= 2b\int_0^{\infty} (b^2t^3 + a^2t + 2abt^2)\, e^{-bt^2}\, dt$$

$$= 2b\int_0^{\infty} b^2t^3 e^{-bt^2}\, dt + 2b\int_0^{\infty} a^2 t e^{-bt^2}\, dt + 2b\int_0^{\infty} 2abt^2 e^{-bt^2}\, dt$$

$$= 2b^3\int_0^{\infty} t^3 e^{-bt^2}\, dt + 2a^2b\int_0^{\infty} t e^{-bt^2}\, dt + 4ab^2\int_0^{\infty} t^2 e^{-bt^2}\, dt$$

0 0 1

$$m_2 = 4ab^2$$

$$\sigma_x^2 = m_2 - (m_1)^2$$
$$= 4ab^2 - (2b^2)^2$$
$$= 4ab^2 - 4b^4$$
$$\sigma_x^2 = 4b^2(a - b^2)$$

Problems :-

1) Consider a R.V $x$ with pdf $f_X(x) = \begin{cases} \frac{1}{K} & \forall\ -2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

where $y = 2x$ find $K$, $E(x)$, $E(y)$, $E(xy)$

Sol:- ① K value :-

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-2}^{3} \frac{1}{K}\,dx = 1$$

$$\frac{1}{K}[x]_{-2}^{3} = 1$$

$$\frac{1}{K}[3-(-2)] = 1 \Rightarrow \frac{5}{K} = 1$$

$$\boxed{K = 5}$$

② $E(x) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

$$= \int_{-2}^{3} x \cdot \frac{1}{5}\,dx = \frac{1}{5}\left[\frac{x^2}{2}\right]_{-2}^{3} = \frac{1}{10}[9-4] = \frac{5}{10} = \frac{1}{2}$$

$$\boxed{E(x) = \frac{1}{2}}$$

③ $E(y) = \int_{-\infty}^{\infty} y \cdot f_X(x)\,dx = \int_{-2}^{3} 2x \cdot f_X(x)\,dx = \int_{-2}^{3} 2x \cdot \frac{1}{5}\,dx$

$$= \frac{2}{5}\left[\frac{x^2}{2}\right]_{-2}^{3} = \frac{1}{5}[9-4] = \frac{1}{5} \times 5$$

$$\boxed{E(y) = 1}$$

(or)

$$E(y) = E(2x) = 2E(x) = 2 \times \frac{1}{2} = 1$$

④ $E(xy) = \int_{-\infty}^{\infty} xy\, f_X(x)\,dx = \int_{-2}^{3} x \cdot 2x \cdot \frac{1}{5}\,dx = \frac{2}{5}\int_{-2}^{3} x^2\,dx$

$$= \frac{2}{5}\left[\frac{x^3}{3}\right]_{-2}^{3} = \frac{2}{15}[27+8] = \frac{2 \times \cancel{35}^{7}}{\cancel{15}_{3}}$$

$$\boxed{E(xy) = \frac{14}{3}}$$

2) Let $X$ be a R.V which take the value 1, 2, 3 with the probability mass function of $\frac{1}{2}, \frac{1}{6}, \frac{1}{2}$ find $3^{rd}$ moment about origin and about mean.

Sol:-

| | | | |
|---|---|---|---|
| $x$ = | 1 | 2 | 3 |
| $P(x)$ = | 1/2 | 1/6 | 1/2 |

① $3^{rd}$ moment about origin $E(x^3) = \sum x^3 \cdot P(x)$

$$= (1)^3 \cdot \frac{1}{2} + (2)^3 \cdot \frac{1}{6} + (3)^3 \cdot \frac{1}{2}$$

$$= \frac{1}{2} + \frac{\cancel{8}^{4}}{\cancel{6}_{3}} + \frac{27}{2} = \frac{3+8+81}{6}$$

$$= \frac{92}{6} = \frac{46}{3} = 15.333$$

② $3^{rd}$ moment about mean

$$E\left[(x-\mu_x)^3\right] \rightarrow \text{skew}$$

$$\mu_x = E(x) = \sum x \cdot p(x) = 1 \cdot \frac{1}{2} + \frac{2}{6} + \frac{3}{2} = \frac{3+2+9}{6}$$

$$= \frac{14}{6} = 2.33$$

$x - \mu_x$

$x=1 \quad 1 - 2.33 = -1.33$

$x=2 \quad 2 - 2.33 = -0.33$

$x=3 \quad 3 - 2.33 = 0.667$

$$E\left[(x-\mu_x)^3\right] = \sum (x-\mu_x)^3 \cdot p(x)$$

$$= (-1.33)^3 \cdot \frac{1}{2} + (-0.33)^3 \cdot \frac{1}{6} + (0.667)^3 \cdot \frac{1}{2}$$

$$= -1.1763 - 0.0059 + 0.1503$$

$$= -1.0319 \approx -1.04$$

3.) The probability density function of a R.V $X$ is given $f_X(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x$ where $x = 0, 1, 2, 3, \ldots \infty$, find MGF of R.V $X$

Sol:-

$$MGF = E(e^{xt}) = \sum_{x=0}^{\infty} e^{xt} p(x)$$

$$= \sum_{x=0}^{\infty} e^{xt} \frac{2}{3}\left(\frac{1}{3}\right)^x$$

$$= \frac{2}{3} \sum_{x=0}^{\infty} e^{xt} \left(\frac{1}{3}\right)^x$$

$$= \frac{2}{3}\sum_{x=0}^{\infty}\left(\frac{e^t}{3}\right)^x \qquad \boxed{\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}}$$

$$= \frac{2}{3}\frac{1}{1-\frac{e^t}{3}}$$

$$= \frac{2}{\cancel{3}}\left[\frac{\cancel{3}}{3-e^t}\right] = \frac{2}{3-e^t}$$

$0^{th}$ moment :- $m_0 = M_x(t)\big|_{t=0} = \frac{2}{3-e^0} = \frac{2}{3-1} = \frac{2}{2} = 1$

$1^{st}$ moment :- $m_1 = \frac{d}{dt}M_x(t)\big|_{t=0}$

4) Let $X$ is continuous R.V, with pdf $f_X(x) = \begin{cases} \frac{x}{12} & \forall\ 1<x<5 \\ 0 & \text{otherwise} \end{cases}$

find MGF, $0^{th}$ moment, $1^{st}$ moment about origin.

Sol:- MGF $= M_x(t) = \int_{-\infty}^{\infty} e^{xt} f_x(x)\,dx = \int_{1}^{5} e^{xt}\cdot\frac{x}{12}\,dx$

$$= \frac{1}{12}\left[\int_1^5 x\cdot e^{xt}\,dx\right]$$

$$= \frac{1}{12}\left\{\left[x\cdot\frac{e^{xt}}{t}\right]_1^5 - \int_1^5 1\cdot\frac{e^{xt}}{t}\,dx\right\}$$

$$= \frac{1}{12}\left[\frac{5\cdot e^{5t}-e^t}{t} - \left[\frac{e^{xt}}{t^2}\right]_1^5\right]$$

$$= \frac{1}{12}\left[\frac{5\cdot e^{5t}-e^t}{t} - \left(\frac{e^{5t}-e^t}{t^2}\right)\right]$$

$$= \frac{1}{12}\left[\frac{5e^{5t}}{t} - \frac{e^t}{t} - \frac{e^{5t}}{t^2} + \frac{e^t}{t^2}\right]$$

$$= \frac{1}{12}\left[\frac{e^{5t}}{t}\left[5-\frac{1}{t}\right] - \frac{e^t}{t}\left[1-\frac{1}{t}\right]\right]$$

$$= \frac{1}{12}\left[\frac{5te^{5t} - te^t - e^{5t} + e^t}{t^2}\right]$$

$$M_x(t) = \frac{1}{12}\left[\frac{e^{5t}(5t-1) - e^t(t-1)}{t^2}\right]$$

5) Find characteristic function of R.V X have density function $f_X(x) = x \quad \forall \; 0 < x < 1$

Sol:- $\phi_X(\omega) = E(e^{j\omega x})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx = \int_0^1 e^{j\omega x} \cdot x \cdot dx$$

$$= \left[ x \cdot \frac{e^{j\omega x}}{j\omega} \right]_0^1 - \int_0^1 1 \cdot \frac{e^{j\omega x}}{j\omega}\,dx .$$

$$= \frac{1 \cdot e^{j\omega} - 0}{j\omega} - \frac{1}{j\omega}\left[ \frac{e^{j\omega x}}{j\omega} \right]_0^1$$

$$\phi_X(\omega) = \frac{e^{j\omega}}{j\omega} + \frac{1}{\omega^2}\left[ e^{j\omega} - 1 \right]$$

6.) A R.V X whose density function is $f_X(x) = b e^{-|ax|}$ where a, b are constant find MGF

Sol:- $MGF = E(e^{xt})$

$$= \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx \Rightarrow \int_{-\infty}^{\infty} e^{xt} \cdot b e^{-|ax|}\,dx$$

$|X| = \pm X$

$-\infty \quad 0 \quad +\infty$

$$= \int_{-\infty}^{0} e^{xt}\, b e^{-a(-x)}\,dx + \int_0^{\infty} e^{xt} \cdot b e^{-ax}\,dx$$

$$= \int_{-\infty}^{0} b \cdot e^{xt} \cdot e^{ax}\,dx + \int_0^{\infty} b \cdot e^{xt} \cdot e^{-ax}\,dx$$

$$= \int_{-\infty}^{0} b \cdot e^{(a+t)x}\,dx + \int_0^{\infty} b \cdot e^{-(a-t)x}\,dx .$$

$$= \left[ b \cdot \frac{e^{(a+t)x}}{a+t} \right]_{-\infty}^{0} + \left[ b \cdot \frac{e^{-(a-t)x}}{-(a-t)} \right]_0^{\infty}$$

$$= \frac{b}{a+t}\left[1 - 0\right] - \frac{b}{(a-t)}\left[0 - 1\right]$$

$$= \frac{b}{a+t} + \frac{b}{a-t} = \frac{ab - bt + ab + bt}{a^2 - t^2} = \frac{2ab}{a^2 - t^2}$$

Transformation problem

3) $y = 2x+3$, A R.V $x$ is uniformly distributed (2,5) find $f_Y(3)$.

Sol:-

$$\boxed{f_Y(y) = f_X(x) \cdot \frac{dx}{dy}}$$

$$f_X(x) = \frac{1}{b-a} = \frac{1}{5-2} = \frac{1}{3}$$

$$f_X(y) = \frac{1}{3} \cdot \frac{dx}{dy}$$

$y = 2x+3$

$dy = 2dx + 0 \Rightarrow \frac{dx}{dy} = \frac{1}{2}$

$$f_X(y) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

4) Find pdf of $Y$ if $f_X(x) = \frac{x}{20}$ $\forall$ $2 < x \leq 5$ where $y = 3x-5$

Sol:-

$$f_Y(y) = f_X(x) \frac{dx}{dy}$$

$$= \frac{x}{20} \cdot \frac{dx}{dy}$$

$y = 3x-5$

$dy = 3dx$

$\frac{dx}{dy} = \frac{1}{3}$

$$= \frac{x}{20} \cdot \frac{1}{3}$$

$$= \frac{x}{60}.$$

Replace $x$ interms of $y$

$Y = 3x - 5$

$x = \frac{y+5}{3}$

$$f_Y(y) = \frac{y+5}{3(60)} = \frac{y+5}{180}$$

5) $f_X(x) = \frac{1}{b} e^{-\frac{(x-a)}{b}}$ MGF, $m_1$

MGF $M_X(t) = E\left[e^{xt}\right] = \int_{-\infty}^{\infty} e^{xt} f_X(x) dx$

$$= \int_a^{\infty} e^{xt} \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$$

$$= \frac{1}{b} \int_a^{\infty} e^{xt} \cdot e^{-x/b} \cdot e^{a/b} dx$$

$$= \frac{1}{b} e^{a/b} \int_a^{\infty} e^{-\left(\frac{1}{b}-t\right)x} dx.$$

$$= \frac{1}{b} e^{a/b} \left[ \frac{e^{-\left(\frac{1}{b}-t\right)x}}{-\left(\frac{1}{b}-t\right)} \right]_a^{\infty}$$

$$= \frac{1}{b} e^{a/b} \left[ \frac{0 - e^{-\left(\frac{1-bt}{b}\right)a}}{-\left(\frac{1-bt}{b}\right)} \right]$$

$M_X(t)$

$$m_1 = E(X) = \int_a^{\infty} x \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} dx.$$